Postulates in Geometry and Philosophy

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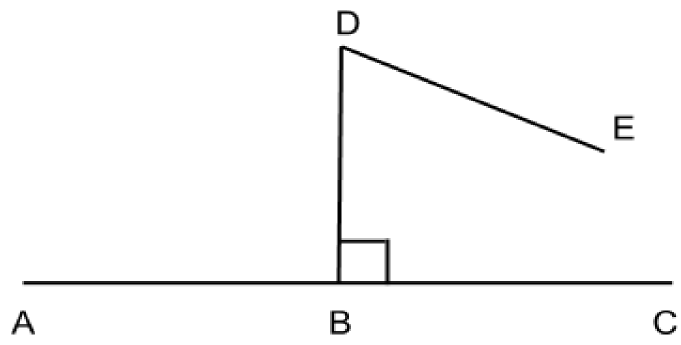
For Plato, mathematics was intimately related to philosophy. Plato reportedly inscribed the command "Let none who is ignorant of geometry enter here" above his Academy's awning. In Plato's *Meno*, the search for the diagonal (and its implicit incommensurability) symbolizes the experience of ἀπορία, or impasse, that is necessary for learning (82B-84D). In the *Theaetetus*, Theaetetus's division of number provides a paradigm for the definition of knowledge (147C - 148B). But what if geometry is itself divided? Lobachevski's *Theory of Parallels* develops a viable non-Euclidean geometry and threatens this salutary relationship between geometry and philosophy. In this paper, I will review Lobachevski's response to Euclid's fifth postulate as a paradigm for understanding how philosophy can relate to its postulates.

Euclid's *Elements* collects much of ancient Greek geometry and number theory into a coherent whole. The book begins with definitions, common notions, and postulates, all of which allow Euclid to prove propositions that follow from those beginnings. The definitions and common notions may be peculiar—for example, a point is defined as "that which has no part"—but they seem unobjectionable enough (Euclid *Elements* 1). The first four postulates are fairly straightforward, granting us abilities like connecting two points with a straight line or declaring that all right angles are equal to each other (Euclid *Elements* 2). However, the fifth and final postulate is stranger than the others. Euclid requests (ἠιτήσθω) the following:

> "That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles." (*Euclid's Elements of*

Geometry 7; *Elements* 2)

The following diagram brings out just such a situation. It depicts three finite line segments, AC, BD, and DE:



BD makes a right angle with AC, but DE makes less than a right angle with BD. When applied to this diagram, Euclid's fifth postulate says that DE, if extended indefinitely, will cut AC on the side of BC. It does look like DE should cut BC. If, when making this diagram, I had extended DE, the software program that I used to make the diagram would have made a line that cut BC. Our experience, intuition, and imagination are ready to grant this request. However, if we question the postulate that our intuition so eagerly wants to accept, we find non-Euclidean geometry.

Euclid's prose and his strategic ordering of the different propositions suggest that he himself was hesitant about the use of this postulate. Euclid avoids using the postulate until the twenty-ninth proposition of his first book; the previous twenty-eight do not use the fifth postulate and are valid without it. Unlike the other postulates, the fifth postulate is a conditional ("if...then") and is relatively long.

These latter differences are superficial. Dr. Halsted provides an alternative to the parallel postulate, Playfair's Postulate, which is both concise and assertive: “Two lines which intersect cannot both be parallel to the same line” (Lobachevski 7). This alternative is ultimately equivalent to Euclid's postulate: it can be shown from this formulation that the lines AC and DE in our diagram will cut each other when extended. Let us put aside the alternative formulation and examine the fifth postulate itself. What makes the fifth postulate uncertain in Euclid, and unacceptable to geometers like Lobachevski? Are all postulates subject to this problem, or is it just this particular postulate?

One possible deficiency of the postulate is that the point of intersection has an indeterminate location that is outside of what is given geometrically. Another possibility is that the necessity of an intersection is not contained within the concept of parallel lines as articulated by Euclid's definition[1]. However, Proclus provides the best answer for why the fifth postulate is ultimately unacceptable.

Geometry, Proclus asserts, cannot accept merely “plausible reasoning” (151). While it is “true and necessary” that “straight lines converge when the right angles are diminished, . . . the conclusion that because they converge more as they are extended farther they will meet at some time is plausible, but not necessary, in the absence of an argument that this is true of straight lines” (150-151). Proclus reminds us that we have reason to consider the Fifth Postulate's emendation of the notion of parallel lines as specious. There are “lines that approach each other indefinitely but never meet,” and while this had only been “ascertained for other species of lines,” namely curved lines, Proclus thought it might be possible to find straight

1 Euclid defines “parallel straight lines” as “straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction” (*Elements* 2).

lines[2] that also approached indefinitely without meeting (151). Until mathematicians had "firmly demonstrated that [such straight lines] meet," reason could ignore imagination's claim to the plausibility of their intersection and attempt to make do without the fifth postulate (151).

When a postulate comes into question, there are several options. If we can replace the postulate with a proof or theorem, then the postulate need not have been requested in the first place. Proclus tried to demonstrate the fifth postulate as a theorem, but mistakenly assumed the conclusion implicitly. Similarly, Saccheri attempted to demonstrate in his book, *Euclid Freed of Every Flaw*, that denying the fifth postulate led to contradictory or absurd results, but his final proof of its absurdity used false reasoning. He assumed that the property "two lines cannot be perpendicular from the same point to a given line and still be the same line" is true in the infinite range—something that would disprove "the hypothesis of the acute angle"—but in fact, this property is "only true in finite ranges" (McCleary 38). Saccheri failed in his mission; ironically, his book became a foundational text of non-Euclidean geometry.

When a postulate has been questioned and when we cannot replace it with a proof or theorem, we can demonstrate what follows from omitting the postulate. Both Saccheri and Lobachevski attempted to do this, but unlike Saccheri, Lobachevski did not hope or expect to find a proof that the fifth postulate was necessary. By denying or putting aside the Fifth Postulate, Lobachevski demonstrates what follows from Euclidean geometry's "uncertainty" about the status and nature of parallel lines and seems to affirm that geometry must only

2 We might understand the straight line as being equivalent between the geometries *individually*, but that *multiple* straight lines put in relation to one another behave differently in the distinct geometries—in Euclidean geometry, line DE cuts BC; in Lobachevskian geometry, line DE might approach but not cut BC.

accept what can be proven to follow from acceptable axioms (13). His Theorem 16, for example, assumes that it is possible that there are multiple non-cutting lines (13-14). Lobachevski retroactively redefines parallel lines as the “boundary lines of the one and the other class of [cutting and not-cutting] lines.”[3] (13) Lobachevski reveals what the immediate consequences of this possibility are and implies that the case where there is only one non-cutting line (as in Euclidean geometry) is a restricted case. Because Euclid's fifth postulate is tantamount to stating that there can only be one non-cutting line, Euclid would find Lobachevski's Theorem 16 logically acceptable but would deny the plausibility or importance of the cases where there is more than one non-cutting line.

In Theorem 22, Lobachevski asserts that the difference between his geometry and Euclid's is fundamental. We have to assume that the sum of the three angles in a triangle is either equal to or less than π radians[4], and that the angle of parallelism is either equal to or less than ½ radians[5]. If we assume that they are equal in each case, we enter ordinary, Euclidean geometry. If we assume that the sum of the angles in a triangle and the angle of parallelism are less than π and ½ π respectively, then we enter Lobachevski's new “Imaginary Geometry” (19).

This bisection implies that there could be more than two geometries. While the fifth postulate's content may seem different from the other requests, we can imagine certain

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3 This new definition is reminiscent of, and likely indebted to, Dedekind's development in number theory of a “cut,” which allows for a rigorous definition of continuity (Dedekind 8-12). Note, however, that Dedekind's use of the word “cut” is not the same as Lobachevski's use of cutting and non-cutting lines.

4 180 degrees.

5 90 degrees, a right angle.

geometers who go further than Lobachevski and remove other postulates[6]. They might ask why it should be the case that all right angles are equal to one another; or they might revoke our ability to draw straight lines or perfect circles. These stubborn geometers would embrace a relativistic[7] plurality of geometries. If you met them, they would ask you which requests you were ready to grant, and tell you what truths followed from those requests. There would be no absolute geometrical truth.

In this world of many geometries, there might even be a geometry where no postulates are allowed, where no requests are granted. No truths whatsoever could emerge from such a geometry. We would be unable to demonstrate Euclid's first proposition—the construction of an equilateral triangle on a given straight line segment—let alone the twenty-ninth (the first proposition to use the parallel postulate). This kind of mathematical skepticism might lead us to believe that we cannot be certain about anything.

Mathematical skepticism, like all skepticism, is not particularly desirable. In mathematics, once we make an assumption, we have hypothetical but certain knowledge about what follows from that assumption. If we deny the fifth postulate, certain strange but true results necessarily follow. In Lobachevskian geometry, the sum of angles in a triangle is less than π; there is an absolute measure of length, a largest triangle (which has an area, *t*, that

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6 In this ahistorical, imaginary illustration, I speak in terms of Euclid's postulates, rather than those of modern geometry. Modern geometry has shown that it is indeed possible to have fewer, more restrictive postulates.

7 I use this word to denote the philosophical implications of non-Euclidean geometry. I understand that Edward Purcell's book *The Crisis of Democratic Theory* contains claims that non-Euclidean geometry inspired the birth of ethical relativism and even nihilism. (See Grant Franks' lecture, “*Everything Aristotle Has Said is Wrong”: On The Authority of Texts and How We Got This Way*).

approaches a finite maximum as its angle sum approaches zero)[8], and a different trigonometry. Alternatively, if we accept the fifth postulate, the Pythagorean theorem is simply true, and there can be neither an absolute measure of length nor a largest triangle.

One of the strange results within Lobachevskian geometry is a new three-dimensional figure called a horosphere[9] (*Non-Euclidean Geometry* 46-57). This figure changes the way we think about the relationship between the two geometries. An equivalent of the fifth postulate can be proven upon the horosphere's surface so that the geometry of the horosphere's surface is equivalent to Euclidean geometry. Therefore, if Lobachevskian geometry is consistent, so is Euclidean geometry, and further, if Euclidean geometry is inconsistent, so is Lobachevskian geometry. We can also prove, using the Klein model, that if Euclidean geometry is consistent, so is Lobachevskian—which would also mean that if Lobachevskian is inconsistent, so is Euclidean geometry (*Non-Euclidean Geometry* 52-55). If we prove the inconsistency of one, the other is also proved inconsistent; if we assume that one is consistent, we must also assume that the other is consistent. The consistency of Euclidean geometry cannot be an argument for dismissing Lobachevskian geometry because it is equivalent to Euclidean geometry with respect to its consistency[10].

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8 Gauss says that "if it could be proved that there can be a rectilineal triangle whose area is greater than any given area, I would be in a position to demonstrate with full rigor the whole of [Euclidean] geometry" (*Non-Euclidean Geometry* 67). For Gauss, such a proof would be equivalent to Postulate Five in Euclidean geometry (albeit as a proof rather than a postulate). He states that while most people would accept that as an axiom, he would not be willing to do so. In Lobachevskian geometry, there is indeed a 'largest triangle,' so that there may not be a rectilineal triangle whose area is greater than any given area. See *Non-Euclidean Geometry* 67-74.

9 The horosphere is constructed by means of a two-dimensional line called the *oricycle* or horocycle, which Lobachevski develops in his Theorem 31 (Lobachevski 30-31).

10 *Non-Euclidean Geometry* makes a more refined distinction on p. 56-7 about the relationship between Lobachevskian solid geometry, Lobachevskian plane geometry, and Euclidean plane geometry. This distinction,

Lobachevskian geometry reminds us that there can only be hypothetical knowledge about the relation of geometrical objects. Its existence implies mathematical relativism and suggests the possible reaction of skepticism. We have also seen that hypothetical certainty about different alternatives is superior to complete ignorance. Still, geometry can no longer be—in Socrates's words—"for the sake of knowing what *is* always" (*Republic* 526E). Rather, it can only be the study of what holds under certain assumptions and conditions, nor can we assume, as we may have been tempted, that the fifth postulate is simply true or false. Euclidean and Lobachevskian geometries are intimately related although they begin from opposite assumptions.

In Plato's *Republic*, when Socrates discusses geometry and dialectic, Socrates asserts that geometry dreams of being but will remain asleep as long as it leaves its hypotheses untouched (533C-D). Geometrical objects like triangles and quadrilaterals, at least as we ordinarily conceive of them, require an unstated assumption about the angle of parallelism. Lobachevski awoke geometry by demonstrating what follows from the denial of the fifth postulate. As "the dialectical way of inquiry proceeds" to awaken, and turn over hypotheses, it will "[destroy all] the hypotheses, to the beginning itself in order to make it secure" (533C–D). We could see the possibility of the continuation of this dialectical motion within mathematics. Other geometers could reject other postulates in turn, leading to various mathematical impasses, so that mathematicians are "really buried in a barbaric bog" (533D).

Eventually, mathematics, like dialectic itself, finds or forces a way out. The way out

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however, does not impact our use of Lobachevski's relation to Euclid's fifth postulate as a paradigm for philosophy's use of postulates.

cannot be quite what we thought it would be, but geometry stands on firmer grounds than when it began. Upon rejecting the other postulates, mathematicians would find similarly unexpected interrelations and consequences. On the other hand, if we decide to return to the ordinary Euclidean geometry that our intuition insists on, we will know its foundations with even greater thoroughness.

The concepts and objects that philosophy is concerned with are even less tangible than lines and figures, but we cannot force ourselves to ignore them. Mathematics shows philosophy the way through its impasses. The basis of philosophy cannot be merely plausible opinions; we cannot leave our ideas and thoughts about philosophical subjects like metaphysics, epistemology, and ethics unconsidered. We can begin our philosophical activity by articulating our opinions as postulates or hypotheses; we can reject, negate, prove, refine, or replace these postulates. This movement from dreaming to awakening—from "dogmatic slumber"[11] to honest, vigorous thinking—might reveal implicit, unexpected relationships between apparently opposite hypotheses. If our original postulates remain viable, we may return to them with a greater understanding of what is implicit within and what follows from those postulates.

Works Cited

Dedekind, Richard. *Essays on the Theory of Numbers*. Mineola: Dover, 1963. Print.

Euclid. *Euclid's Elements of Geometry*. Greek compiled by J. L. Heiberg; translated by Richard Fitzpatrick. Lulu Books, 2007. Print.

11 This is Kant's famous remark about his relationship to Hume (Kant 10), one which we might compare to our understanding of Euclid and Lobachevski.

Euclid. *Elements*. Translated by Sir Thomas L. Heath. Santa Fe: Green Lion Press, 2007. Print.

Franks, Grant. *"Everything Aristotle Has Said Is Wrong": The Authority of Texts and How We Got This Way."* Lecture. Website: http://home.earthlink.net/~grantfranks/Everything%20Aristotle%20Has%20.htm.

Kant, Immanuel. *Prolegomena to Any Future Metaphysics*, translated and edited by Gary Hatfield, revised edition (Cambridge: Cambridge University Press, 2004). Print.

Lobachevski, Nicholas. *The Theory of Parallels*. Translated by Dr. George Halsted. Annapolis: St. John's College Press, 2012. Print.

McCleary, John. *Geometry from a Differentiable Viewpoint*. Cambridge: Cambridge University Press, 1997. Print.

Non-Euclidean Geometry. Annapolis: St. John's College Press, 2012. Print.

Plato. *Meno*. Tranlated by Anastaplo and Berns. Newburyport: Focus, 1998. Print.

Plato. *Republic*. Translated by Allan Bloom. New York: Basic Books, 1991. Print.

Plato. *Theaetetus*. Translated by Joe Sachs. Newburyport: Focus, 2004. Print.

Proclus. *A Commentary on the First Book of Euclid's Elements*. Translated by Glenn R. Morrow. Princeton: Princeton University Press, 1992. Print.