The Mystery of Points, the Viciousness of Circles, and the Controversy of Lines

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Geometry is one of the most ancient fields of mathematics. The written works of the geometric studies of the Hindu, Chinese, Babylonian, and Egyptian civilizations extend back at least to 1700 BC. As a field of mathematics with such a long history, geometry offers a chance to examine a field of study through great changes and bitter controversies. A study of the foundations of geometry presents an unusual opportunity for students to learn about the process of mathematics through the historical progression of ideas and the questioning of one's most basic assumptions. In an examination of the hope of early mathematicians to formulate clearly statements that provide the foundations of geometry, students of mathematics can find that the controversial is necessary, the absent is essential, and the attempt to fill in every gap is deadly.

The early mathematical work of ancient civilizations tended to focus on approximation and applicability rather than on exactness and proof. The studies of mathematics in Greece, first under Pythagoras and then under Plato, led in a different direction. Proclus Diadochus in his *Commentary on the First Book of Euclid's "Elements of Geometry"* states, "Pythagoras changed the study of geometry, giving it the form of a liberal discipline, seeking its first principles in ultimate ideas, and investigating its theorems abstractly in a purely intellectual way" (6). Euclid continued on this path in what one might think of as the extension of the project of Pythagoras. In his *Elements*, he tried rigorously to generate all the known theorems of plane geometry from five basic statements, which he called "postulates," and a handful of definitions. His attempt to develop geometry and, at the same time, number theory from basic postulates, or axioms, as we would now call them, was perhaps the most important moment in the recorded history of mathematics. Though Euclid's axioms eventually proved inadequate to his task, his method was sound and has been the source of a great deal of the work of mathematicians in geometry and in all the other fields of mathematics until the present day.

Oddly enough, what finally proved the source of Euclid's failure was not the greatest controversy arising from *Elements*. That distinction belongs to his infamous Postulate V:

> If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines if produced indefinitely, meet on that side on which are the angles less than the two right angles. (155)

At over four times the word count of the longest of his other postulates and lacking any of their intuitive charm, there is no wonder that Proclus said of Postulate V: "[It] ought even to be struck out of the Postulates altogether." (13) Proclus thought Postulate V "[was] a theorem" and that "its obvious character [did] not appear independently of proof" (13). Proclus was not alone in his discomfort with this seemingly not-self-evident-enough statement. Generations of talented mathematicians followed in attempt after obsessive attempt to prove Postulate V from Euclid's other four and their results. Adrien-Marie Lagrange made many of these attempts, struggling for twenty-nine years of his life with the task (Greenberg 23).

Postulate V's proof was elusive for a reason; it did not exist. This axiom was independent of all the others; they could stand or fall with or without it. However, Euclid's work was still not without errors. By focusing on the stain on the floor where Postulate V lay, mathematicians failed to see the collapsing floorboards in other places, areas of danger that Euclid had merely leapt across with his many unstated assumptions. Many of these assumptions were far more self-evident than Postulate V, but their very obviousness hindered their discovery. Euclid neglected to state the many properties that we must know about how points on a line lie between one another; he did not state that lines were meant to progress continuously from point to point; he never even stated that points or lines exist.

If, in some ways, the weakness with Euclid's axioms was in leaving too much unsaid, then the weakness of his definitions was in saying too much. Euclid tried to clearly define each geometric term that he used. Euclid's first definition in *Elements* was "a point is that which has no part," a definition that apparently compounds its sin of vagueness with that of uselessness; Euclid never refers to the definition again (153). In his abandonment of his first definition, Euclid was on surer footing than he was in its initial statement. Over two millennia later, Karl Popper gave one explanation why: "For a definition cannot establish the meaning of a term any more than a logical derivation can establish the truth of a statement; both can only shift this problem back" (17). Just as some axioms must remain unproven, some terms must remain undefined. If we try to write clear definitions for every term that we use, we find ourselves in danger of eventually defining at least some of the words in our finite list circularly, in terms of themselves, or of writing an infinitely long dictionary, careful to use only the newest, best, most Dr. Seuss-like words once we run out of those more pedestrian and less exciting words we all know. If our words have meaning, it cannot lie solely in their definitions.

The struggle to repair Euclid's axiomatic treatment of geometry by filling in the gaps of his unstated assumptions yielded unexpected benefits. Once the lack of dependence of Postulate V on Euclid's other axioms was discovered, the fruitless attempts to find its proof ended. This discovery eventually led to a new understanding of what the axiomatic method could mean for geometry. If Postulate V was independent of the other axioms, what sorts of geometric worlds might its logical negation allow us to describe? The as-near-to-simultaneous-as-relativity-and-history-will-allow discovery of non-Euclidean geometry by János Bolyai and Carl Freidrich

Gauss in the nineteenth century was the result (Greenberg 239-245). Through an accident of planetary size, humanity developed a system of thinking about utterly flat planes and utterly straight lines while living on a roughly round surface. The discovery of non-Euclidean geometry finally allowed mathematicians to open up their study of this familiar round surface along with other, even stranger, surfaces. Then the expansion of Euclid's axioms led the way to examine what might happen if they were to be eliminated or changed.

The discovery of non-Euclidean geometry was not the only value of the work of Euclid's intellectual descendants. The use of terms without definition allowed the users of geometry to wonder in what different ways one might interpret these undefined terms. With all but a few axioms removed, simple versions of geometry, such as incidence geometry in which even basic terms, like "between" hold no meaning, arise offering many interpretations. Even in systems with more axioms, different interpretations reveal themselves. The same systems can be used to describe worlds of different dimensions, as we see in a two-dimensional hologram that reveals a three-dimensional image. In fact, some new theories of physics "predict that the number of dimensions in reality could be a matter of perspective ... obeying one set of laws ... in three dimensions or equivalently, as obeying a different set of laws that operates in two dimensions" (Maldacena 147). The value of such varying interpretations is that in providing a different point of view, they might make simpler the solution of some difficult problems of physics, such as the search for a quantum theory of gravity (Maldacena 148).

Through an examination of the axiomatic method and its historical controversies, students of geometry can be spurred to open their minds to new ways of envisioning their studies as were the mathematicians of the past. Axioms that initially strike us as non-intuitive or bizarre intractably resist removal, yet when we attempt to remove them, we discover that we effectively warp the flat pages on which they were written. In searching to justify these axioms, students can learn to imagine a world without them. Fundamental terms prove impossible to define clearly, yet this very difficulty allows young mathematicians to consider different explanations for what we might mean by them. In striving to do the impossible, we force ourselves to see new sights, think new thoughts, and discover new worlds.

Works Cited

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